A New Construction of Quasi-solvable Quantum Many-body Systems of Deformed Calogero–Sutherland Type

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Abstract

We make a new multivariate generalization of the type A monomial space of a single variable. It is different from the previously introduced type A space of several variables which is an $\mathfrak{sl}(M+1)$ module, and we thus call it type A′. We construct the most general quasi-solvable operator of (at most) second-order which preserves the type A′ space. Investigating directly the condition under which the type A′ operators can be transformed to Schrödinger operators, we obtain the complete list of the type A′ quasi-solvable quantum many-body systems. In particular, we find new quasi-solvable models of deformed Calogero–Sutherland type which are different from the Inozemtsev systems. We also examine a new multivariate generalization of the type C monomial space based on the type A′ scheme.

Key words: quantum many-body problem, quasi(-exact) solvability, Calogero–Sutherland models
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1 Introduction

It is widely known that most of the quasi-exactly solvable quantum one-body Hamiltonians, for which we can obtain a part of the exact eigenvalues and eigenfunctions in closed form [1,2], have the underlying $\mathfrak{sl}(2)$ structure first discovered in Ref. [3]. Recently, it was found in Ref. [4] that the well-known

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exactly solvable M-body Calogero–Sutherland models [5,6] have a similar Lie-algebraic structure of $\mathfrak{sl}(M+1)$. After the several new discoveries of quasi-exactly solvable Hamiltonians having the same $\mathfrak{sl}(M+1)$ Lie-algebraic structure [7,8,9,10], the complete list of the quantum many-body systems which admit the $\mathfrak{sl}(M+1)$ structure was obtained in Ref. [11]. It turns out that all of them are of Inozemtsev type, which were originally found to be classically integrable [12,13,14]. The common feature of all the models which have an underlying Lie-algebraic structure is that they leave a finite dimensional module of the Lie algebra invariant.

On the other hand, Post and Turbiner studied a classification of linear differential operators of a single variable which have a finite dimensional invariant subspace spanned by monomials [15]. According to the results in Ref. [15] combined with the discussion in Ref. [16], there are essentially three different spaces of monomial type preserved by second-order linear differential operators, except for a few special cases. Later, they were dubbed type A, B, and C, respectively, according to the corresponding types of $\mathcal{N}$-fold supersymmetry [17]. The type A space corresponds to the $\mathfrak{sl}(2)$ module investigated in Ref. [3] and second-order linear differential operators preserving it are expressed as quadratic forms of the $\mathfrak{sl}(2)$ generators represented by first-order linear differential operators. The $\mathfrak{sl}(M+1)$ module preserved by the quantum Inozemtsev systems was then regarded as a natural generalization of the type A monomial space of a single variable to several variables [18].

However, the other two spaces, type B and C, are not Lie-algebraic modules and linear differential operators preserving the type B or C spaces are not given through the universal enveloping algebra of any Lie algebras. Recently, we have successfully generalized the type C monomial space of a single variable to several variables and constructed the most general second-order linear differential operator preserving it [18]. Throughout our experiences in studying the latter problems and searching for a natural generalization of the type B monomial space of a single variable to several variables (cf. Section 8 in Ref. [18]), we are convinced that there are much more varieties of monomial type space of several variables than those of a single variable which can be preserved by linear differential operators if we do not restrict ourselves to study such an operator that admits an underlying Lie-algebraic structure.

In this article, we show that by making another generalization of the type A monomial space of a single variable, which is different from the $\mathfrak{sl}(M+1)$ module investigated fully in Ref. [11], we obtain a new family of quasi-solvable quantum many-body systems of deformed Calogero–Sutherland type. The obtained models turn to have in general M-body interaction terms and are thus different from the Inozemtsev systems.

The article is organized as follows. In the next section, we summarize some

definitions related to the concept of quasi-solvability in order to avoid ambiguity. In Section 3, we introduce a new generalization of the type A monomial space of a single variable to several variables, which we shall call type A′, and briefly discuss some important property of the type A′ space such as the invariance under linear transformations. In Section 4, we construct the most general (at most) second-order linear differential operator which preserves the type A′ space. In Section 5, we examine the condition under which the general type A′ quasi-solvable operators can be transformed to Schrödinger operators. Utilizing the invariance under the linear transformations, we fully classify the possible quantum Hamiltonians preserving the type A′ space in Section 6. In Section 7, we make a new multivariate generalization of the type C monomial space of a single variable, which we shall call type C′, based on the type A′ space. We then construct the most general type C′ quasi-solvable operator of (at most) second-order as well as the type C′ gauged Hamiltonian. Finally in Section 8, we summarize and discuss the obtained results in combination with the results in the type A and C cases. Some useful formulas are summarized in Appendix A.

2 Definition

First of all, we shall give the definition of quasi-solvability and some notions of its special cases based on Refs. [11,19]. A linear differential operator H of several variables $q = (q_1, \dots, q_M)$ is said to be *quasi-solvable* if it preserves a finite dimensional functional space $\mathcal{V}_\mathcal{N}$ whose basis admits an analytic expression $\phi_i(q)$ in closed form [2]:

$$H\mathcal{V}_\mathcal{N} \subset \mathcal{V}_\mathcal{N}, \quad \dim \mathcal{V}_\mathcal{N} = n(\mathcal{N}) < \infty, \quad \mathcal{V}_\mathcal{N} = \Big\langle \phi_1(q), \dots, \phi_{n(\mathcal{N})}(q) \Big\rangle. \tag{2.1}$$

An immediate consequence of the above definition of quasi-solvability is that, since we can calculate finite dimensional matrix elements $\mathbf{S}_{k,l}$ defined by,

$$H\phi_k = \sum_{l=1}^{n(\mathcal{N})} \mathbf{S}_{k,l}\phi_l \qquad \big(k = 1, \dots, n(\mathcal{N})\big), \tag{2.2}$$

we can diagonalize the operator H and obtain its spectra in the space $\mathcal{V}_\mathcal{N}$, at least, algebraically. Furthermore, if the space $\mathcal{V}_\mathcal{N}$ is a subspace of a Hilbert space $L^2(S)$ $(S \subset \mathbb{R}^M)$ on which the operator H is naturally defined, the calculable spectra and the corresponding vectors in $\mathcal{V}_\mathcal{N}$ give the *exact* eigenvalues and eigenfunctions of H, respectively. In this case, the operator H is

[2] The latter restriction has been sometimes missed in the literature. Without it, however, arbitrary linear operators would be quasi-solvable unless their spectrum is empty.

said to be *quasi-exactly solvable* (on S)[3]. Otherwise, the calculable spectra and the corresponding vectors in $\mathcal{V}_{\mathcal{N}}$ only give *local* solutions of the characteristic equation of H. This important difference has been sometimes missed in the literature.

A quasi-solvable operator H of several variables is said to be *solvable* if it preserves an infinite flag of finite dimensional functional spaces $\mathcal{V}_{\mathcal{N}}$,

$$\mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_{\mathcal{N}} \subset \cdots , \tag{2.3}$$

whose bases admit analytic expressions in closed form, that is,

$$H\mathcal{V}_{\mathcal{N}} \subset \mathcal{V}_{\mathcal{N}}, \quad \dim \mathcal{V}_{\mathcal{N}} = n(\mathcal{N}) < \infty, \quad \mathcal{V}_{\mathcal{N}} = \left\langle \phi_1(q), \dots, \phi_{n(\mathcal{N})}(q) \right\rangle, \tag{2.4}$$

for $\mathcal{N} = 1, 2, 3, \dots$. Furthermore, if the sequence of the spaces (2.3) defined on $S \subset \mathbb{R}^M$ satisfies,

$$\overline{\mathcal{V}_{\mathcal{N}}(S)} \to L^2(S) \qquad (\mathcal{N} \to \infty), \tag{2.5}$$

the operator H is said to be *exactly solvable* (on S).

3 A New Generalization of the Type A Monomial Space

In this article, we shall consider a quantum mechanical system of M identical particles on a line. The Hamiltonian is thus given by

$$H = -\frac{1}{2}\sum_{i=1}^{M} \frac{\partial^2}{\partial q_i^2} + V(q_1, \dots, q_M), \tag{3.1}$$

where the potential has permutation symmetry:

$$V(\dots, q_i, \dots, q_j, \dots) = V(\dots, q_j, \dots, q_i, \dots) \qquad \forall i \neq j\,. \tag{3.2}$$

To construct a quasi-solvable operator of the form (3.1), we shall follow the three steps after Ref. [11], namely, i) a *gauge* transformation on the Hamiltonian (3.1):

$$\tilde{H} = \mathrm{e}^{\mathcal{W}(q)} H \mathrm{e}^{-\mathcal{W}(q)}, \tag{3.3}$$

[3] A domain S is not necessarily a subspace of the real space $\mathbb{R}^M$ if the operator under consideration is non-Hermitian. Indeed, a couple of quasi-solvable one-body Hamiltonians which are not quasi-exactly solvable on any subspaces of the real space $\mathbb{R}$ are shown to be quasi-exactly solvable on the subspaces of the *complex* space $\mathbb{C}$ incorporating with the $\mathcal{PT}$-symmetric boundary conditions [20,21].

ii) a change of variables from q_i to z_i by a function z of a single variable:

$$z_i = z(q_i), \tag{3.4}$$

and iii) the introduction of the elementary symmetric polynomials of z_i defined by,

$$\sigma_k(z) = \sum_{i_1<\cdots<i_k}^{M} z_{i_1} \ldots z_{i_k} \quad (k = 1, \ldots, M), \qquad \sigma_0 \equiv 1\,. \tag{3.5}$$

Due to the permutation symmetry of the original Hamiltonian (3.1), the gauged Hamiltonian (3.3) can be completely expressed in terms of the elementary symmetric polynomials (3.5). In this article, a second-order linear differential operator is called a *gauged Hamiltonian* if it can be transformed to a Schrödinger operator by means of a combination of gauge transformations and change of variables.

The next task is to choose a vector space to be preserved by the gauged Hamiltonian (3.3). The type A monomial space of a single variable z is defined by [17]

$$\tilde{\mathcal{V}}^{(A)}_{\mathcal{N}} = \left\langle 1, z, \ldots, z^{\mathcal{N}-1} \right\rangle. \tag{3.6}$$

It is an $\mathfrak{sl}(2)$ module and the foundation of the $\mathfrak{sl}(2)$ construction of quasi-solvable models in Ref. [3]. In our previous paper [18], we identified the following space as a generalization of the type A space of a single variable to several variables:

$$\tilde{\mathcal{V}}^{(A)}_{\mathcal{N};M} = \left\langle \sigma_1^{n_1} \ldots \sigma_M^{n_M} \;\middle|\; n_i \in \mathbb{Z}_{\geq 0},\ \sum_{i=1}^{M} n_i \leq \mathcal{N} - 1 \right\rangle. \tag{3.7}$$

It is indeed a natural generalization since it provides an $\mathfrak{sl}(M+1)$ module for each M and is the foundation of the $\mathfrak{sl}(M+1)$ construction of quasi-solvable quantum many-body systems in Ref. [11]. However, the spaces (3.6) and (3.7) have a different character, that is, the elements of the latter space of multi-variable are not necessarily polynomials of degree less than $\mathcal{N}$ in the variables z_i, in contrast to the former space of a single variable. Hence, another natural generalization would be such that any element of a generalized space is a polynomial of degree less than $\mathcal{N}$ in z_i. Each elementary symmetric polynomial σ_k is a polynomial of degree k in z_i and thus the latter generalization can be realized by the following vector space:

$$\tilde{\mathcal{V}}^{(A')}_{\mathcal{N};M} = \left\langle \sigma_1^{n_1} \ldots \sigma_M^{n_M} \;\middle|\; n_i \in \mathbb{Z}_{\geq 0},\ \sum_{i=1}^{M} i n_i \leq \mathcal{N} - 1 \right\rangle. \tag{3.8}$$

Obviously, the space (3.8) also reduces to the single-variable type A space (3.6) when $M = 1$, but is different from Eq. (3.7). We thus call the space

(3.8) *type A′*. In this article, we investigate linear differential operators which preserve the type A′ space.

In contrast to the fact that the type A space (3.7) is invariant under the $GL(2,\mathbb{R})$ linear fractional transformation [11]:

$$\tilde{\mathcal{V}}_{\mathcal{N}}^{(A)}[z] \mapsto \prod_{i=1}^{M} (\gamma z_i + \delta)^{\mathcal{N}-1} \tilde{\mathcal{V}}_{\mathcal{N}}^{(A)}[\hat{z}] = \tilde{\mathcal{V}}_{\mathcal{N}}^{(A)}[z], \tag{3.9}$$

induced by

$$z_i \mapsto \hat{z}_i = \frac{\alpha z_i + \beta}{\gamma z_i + \delta} \qquad (\alpha, \beta, \gamma, \delta \in \mathbb{R}; \Delta \equiv \alpha\delta - \beta\gamma \neq 0), \tag{3.10}$$

the type A′ space (3.8) does not have the full $GL(2,\mathbb{R})$ invariance for $M > 1$. It is easily read from the transformation of the elementary symmetric polynomials (3.5) under the special projective transformation $z_i \mapsto \hat{z}_i = 1/z_i$:

$$\sigma_k(z) \mapsto \sigma_k(z^{-1}) = \sigma_{M-k}(z)\sigma_M(z)^{-1}. \tag{3.11}$$

Hence, the special projective transformation interchanges the role of the variables σ_k and σ_{M-k} in the space (3.8) and cannot keep the condition $\sum_{i=1}^{M} i n_i \leq \mathcal{N} - 1$ unchanged. However, the other elements of the $GL(2,\mathbb{R})$ transformations (3.10), namely, the dilatation ($\hat{z}_i = \alpha z_i$) and the translation ($\hat{z}_i = z_i + \beta$), preserve the type A′ space (3.8). The former is trivial while the latter is understood from the transformation of σ_k under the translation:

$$\sigma_k(z) \mapsto \sigma_k(z + \beta) = \sum_{l=0}^{k} \beta^{k-l} C_l(M) \sigma_l(z), \tag{3.12}$$

where C_l are constants depending on M. We now easily see that the latter transformation indeed preserves all the elements of (3.8) within the space.

4 Construction of Quasi-solvable Operators

In this section, we shall construct the general quasi-solvable operators of (at most) second-order leaving the type A′ space (3.8) invariant. The set of linearly independent first-order differential operators preserving the type A′ space is given by,

$$F_{\{m_i\}_k,k} \equiv \prod_{i=1}^{M} \sigma_i^{m_i} \frac{\partial}{\partial \sigma_k} \quad (k = 1, \dots, M), \tag{4.1a}$$

$$F_{10} \equiv \sigma_1 \left(\mathcal{N} - 1 - \sum_{k=1}^{M} k \sigma_k \frac{\partial}{\partial \sigma_k} \right). \tag{4.1b}$$

In Eq. (4.1a), $\{m_i\}_k$ is an abbreviation of the set of M non-negative integers defined by

$$\{m_i\}_k \equiv \left\{ m_1, \dots, m_M \,\middle|\, m_i \in \mathbb{Z}_{\geq 0},\ \sum_{i=1}^{M} i m_i \leq k \right\}. \tag{4.2}$$

In the single variable case ($M = 1$), the set of differential operators (4.1) consists of

$$F_{\{0\}_1,1} = \frac{\partial}{\partial\sigma_1}, \quad F_{\{1\}_1,1} = \sigma_1 \frac{\partial}{\partial\sigma_1}, \quad F_{10} = \sigma_1 \left(\mathcal{N} - 1 - \sigma_1 \frac{\partial}{\partial\sigma_1} \right), \tag{4.3}$$

and hence is essentially the same as the $\mathfrak{sl}(2)$ generators. It is as expected since the single-variable type A′ space is nothing but the $\mathfrak{sl}(2)$ module (3.6). In the case of two variables ($M = 2$), the set of differential operators (4.1) is composed of

$$\begin{aligned} &F_{\{00\}_1,1} = \frac{\partial}{\partial\sigma_1}, \quad F_{\{10\}_1,1} = \sigma_1 \frac{\partial}{\partial\sigma_1}, \quad F_{10} = \sigma_1 \left(\mathcal{N} - 1 - \sum_{k=1}^{2} k\sigma_k \frac{\partial}{\partial\sigma_k} \right), \\ &F_{\{01\}_2,2} = \sigma_2 \frac{\partial}{\partial\sigma_2}, \quad F_{\{m_1 0\}_2,2} = \sigma_1^{m_1} \frac{\partial}{\partial\sigma_2} \quad (m_1 = 0, 1, 2), \end{aligned} \tag{4.4}$$

and is essentially the same as the generators of $\mathfrak{gl}(2) \ltimes \mathbb{R}^3$ in Ref. [19]. Actually, the two-variable type A′ space (3.8) coincides with the $\mathfrak{gl}(2) \ltimes \mathbb{R}^3$ module investigated in Ref. [19]. In this sense, the type A′ provides a natural extension of that Lie-algebraic scheme of two variables to arbitrary number of variables.

The set of linearly independent second-order differential operators leaving the type A′ space invariant is as follows:

$$F_{\{m_i\}_{k+l},kl} \equiv \prod_{i=1}^{M} \sigma_i^{m_i} \frac{\partial^2}{\partial\sigma_k \partial\sigma_l} \quad (k, l = 1, \dots, M;\ k \geq l), \tag{4.5}$$

$$F_{10} F_{\{\overline{m}_i\}_k,k} = \sigma_1 \left(\mathcal{N} - 1 - \sum_{l=1}^{M} l\sigma_l \frac{\partial}{\partial\sigma_l} \right) \prod_{i=1}^{M} \sigma_i^{\overline{m}_i} \frac{\partial}{\partial\sigma_k} \quad (k = 1, \dots, M), \tag{4.6}$$

$$F_{10} F_{10} = \sigma_1^2 \left(\mathcal{N} - 2 - \sum_{k=1}^{M} k\sigma_k \frac{\partial}{\partial\sigma_k} \right) \left(\mathcal{N} - 1 - \sum_{l=1}^{M} l\sigma_l \frac{\partial}{\partial\sigma_l} \right), \tag{4.7}$$

$$F_{20,00} \equiv \sigma_2 \left(\mathcal{N} - 2 - \sum_{k=1}^{M} k\sigma_k \frac{\partial}{\partial\sigma_k} \right) \left(\mathcal{N} - 1 - \sum_{l=1}^{M} l\sigma_l \frac{\partial}{\partial\sigma_l} \right), \tag{4.8}$$

where in Eq. (4.6) $\{\overline{m}_i\}_k$ is an abbreviation of the set of M non-negative integers defined by

$$\{\overline{m}_i\}_k \equiv \left\{ \overline{m}_1, \dots, \overline{m}_M \,\middle|\, \overline{m}_i \in \mathbb{Z}_{\geq 0},\ \sum_{i=1}^{M} i \overline{m}_i = k \right\}. \tag{4.9}$$

Several remarks are in order. First of all, it is important to note that the set of the quadratic form of the first-order operators $F_{\{m_i\}_k,k}$ in Eq. (4.1a) cannot exhaust the second-order operators of the form given by Eq. (4.5) when the number of the variables is greater than two ($M > 2$). In the case of $M = 3$, for instance, the following operator which is an element of Eq. (4.5)

$$F_{\{020\}_4,31} = \sigma_2^2 \frac{\partial^2}{\partial\sigma_3 \partial\sigma_1}, \tag{4.10}$$

cannot be represented by any quadratic combination of the first-order operators in Eq. (4.1a). Second, even when the number of the variables is two ($M = 2$) where the type A′ space (3.8) provides a $\mathfrak{gl}(2) \ltimes \mathbb{R}^3$ module, there exist higher-order operators preserving the type A′ space which cannot be expressed as a polynomial in the first-order operators (4.4), such as Eq. (4.8). In this respect, see also Ref. [22]. Third, the operators $F_{10}F_{\{m_i\}_k,k}$ with $\sum_{i=1}^{M} im_i < k$ are represented by linear combinations of the other operators (4.1)–(4.8) as

$$\begin{aligned} F_{10}F_{\{m_i\}_k,k} &= \left(\mathcal{N} - 1 - \sum_{l=1}^{M} lm_l\right)\sigma_1 \prod_{i=1}^{M} \sigma_i^{m_i} \frac{\partial}{\partial\sigma_k} - \sum_{l=1}^{M} l\sigma_1\sigma_l \prod_{i=1}^{M} \sigma_i^{m_i} \frac{\partial^2}{\partial\sigma_k \partial\sigma_l} \\ &= \left(\mathcal{N} - 1 - \sum_{l=1}^{M} lm_l\right) F_{\{m'_i\}_k,k} - \sum_{l=1}^{M} lF_{\{m''_i\}_{k+l},kl}\,, \end{aligned} \tag{4.11}$$

where $\{m'_i\}_k$ is obtained from $\{m_i\}_k$ by $m'_i = m_i + \delta_{i,1}$ while $\{m''_i\}_{k+l}$ is from $\{m_i\}_k$ by $m''_i = m_i + \delta_{i,1} + \delta_{i,l}$. The restriction $\sum_{i=1}^{M} im_i < k$ ensures that $\sum_{i=1}^{M} im'_i \leq k$ and $\sum_{i=1}^{M} im''_i \leq k + l$. Thus, $F_{\{m'_i\}_k,k}$ and $F_{\{m''_i\}_{k+l},kl}$ are in fact members of the operators in Eqs. (4.1a) and (4.5), respectively. Hence, the operators $F_{10}F_{\{m_i\}_k,k}$ with $\sum_{i=1}^{M} im_i < k$ are linearly dependent on the other operators (4.1)–(4.8). That is why in Eq. (4.6) we restrict the values of the set $\{m_i\}$ to the one given by Eq. (4.9). Furthermore, the operators of the form $F_{\{\overline{m}_i\}_k,k}F_{10}$ are also linearly dependent on the set of operators (4.1)–(4.8) since the anti-commutator of $F_{\{\overline{m}_i\}_k,k}$ and F_{10} reads

$$\left[F_{\{\overline{m}_i\}_k,k}, F_{10}\right] = \delta_{k,1}F_{10}. \tag{4.12}$$

Therefore, the most general quasi-solvable operator of (at most) second-order which preserves the type A′ space (3.8) is given by the linear combination of the operators (4.1)–(4.8):

$$\begin{aligned} \tilde{\mathcal{H}}_{\mathcal{N}}^{(A')} = &- \sum_{k\geq l}^{M} \sum_{\{m_i\}_{k+l}} A_{\{m_i\}_{k+l},kl} F_{\{m_i\}_{k+l},kl} - \sum_{k=1}^{M} \sum_{\{\overline{m}_i\}_k} A_{10,\{\overline{m}_i\}_k,k} F_{10} F_{\{\overline{m}_i\}_k,k} \\ &- A_{10,10}F_{10}F_{10} - A_{20,00}F_{20,00} \\ &+ \sum_{k=1}^{M} \sum_{\{m_i\}_k} B_{\{m_i\}_k,k} F_{\{m_i\}_k,k} + B_{10}F_{10} - c_0, \end{aligned} \tag{4.13}$$

where the coefficients A, B with indices and c_0 are real constants and the summation over the set $\{m_i\}_k$ etc. is understood to take all the possible set of values $\{m_1, \dots, m_M\}$ indicated in Eqs. (4.2) and (4.9). In terms of the variables σ, the operator $\tilde{\mathcal{H}}_{\mathcal{N}}^{(A')}$ is expressed as

$$\begin{aligned}\tilde{\mathcal{H}}_{\mathcal{N}}^{(A')} = & -\sum_{k,l=1}^{M}\Big[\boldsymbol{A}_0(\sigma)kl\sigma_k\sigma_l + \boldsymbol{A}_{kl}(\sigma)\Big]\frac{\partial^2}{\partial\sigma_k\partial\sigma_l} \\ & + \sum_{k=1}^{M}\Big[\boldsymbol{B}_0(\sigma)k\sigma_k - \boldsymbol{B}_k(\sigma)\Big]\frac{\partial}{\partial\sigma_k} - \boldsymbol{C}(\sigma),\end{aligned} \tag{4.14}$$

where $\boldsymbol{A}_0$, $\boldsymbol{A}_{kl}$, $\boldsymbol{B}_0$, $\boldsymbol{B}_k$, and $\boldsymbol{C}$ are polynomials of several variables given by

$$\boldsymbol{A}_0(\sigma) = A_{10,10}\sigma_1^2 + A_{20,00}\sigma_2, \tag{4.15a}$$

$$\boldsymbol{A}_{kl}(\sigma) = \sum_{\{m_i\}_{k+l}} A_{\{m_i\}_{k+l},kl}\prod_{i=1}^{M}\sigma_i^{m_i} - \sum_{\{\overline{m}_i\}_k} A_{10,\{\overline{m}_i\}_k,k}l\sigma_1\sigma_l\prod_{i=1}^{M}\sigma_i^{\overline{m}_i}, \tag{4.15b}$$

$$\boldsymbol{B}_0(\sigma) = (2\mathcal{N} - k - 3)\boldsymbol{A}_0(\sigma) - B_{10}\sigma_1, \tag{4.15c}$$

$$\boldsymbol{B}_k(\sigma) = (\mathcal{N} - k - 1)\sum_{\{\overline{m}_i\}_k} A_{10,\{\overline{m}_i\}_k,k}\sigma_1\prod_{i=1}^{M}\sigma_i^{\overline{m}_i} - \sum_{\{m_i\}_k} B_{\{m_i\}_k,k}\prod_{i=1}^{M}\sigma_i^{m_i}, \tag{4.15d}$$

$$\boldsymbol{C}(\sigma) = (\mathcal{N} - 1)(\mathcal{N} - 2)\boldsymbol{A}_0(\sigma) - (\mathcal{N} - 1)B_{10}\sigma_1 + c_0. \tag{4.15e}$$

Among the set of the operators (4.1)–(4.8), $F_{\{m_i\}_k,k}$ and $F_{\{m_i\}_{k+l},kl}$ preserve the type A′ space (3.8) for *arbitrary* natural number $\mathcal{N}$. Therefore, the operator $\tilde{\mathcal{H}}_{\mathcal{N}}^{(A')}$ is not only quasi-solvable but also solvable if

$$A_{10,\{\overline{m}_i\}_k,k} = A_{10,10} = A_{20,00} = B_{10} = 0. \tag{4.16}$$

5 Extraction of Schrödinger Operators

In the preceding section, we have constructed the most general quasi-solvable second-order operator $\tilde{\mathcal{H}}_{\mathcal{N}}^{(A')}$ preserving the type A′ space (3.8). By applying a similarity transformation on $\tilde{\mathcal{H}}_{\mathcal{N}}^{(A')}$ and a change of variables, we may obtain a family of quasi-solvable operators of a desired form. However, second-order linear differential operators of several variables are in general not gauged Hamiltonians, that is, they cannot be always transformed to Schrödinger operators. This fact is one of the most obstacles in constructing quasi-solvable *quantum* many-body systems. Recently in Refs. [11,18,23], it was shown that the amount of the difficulty can be significantly reduced by considering the underlying symmetry of the invariant space of quasi-solvable operators. In the present case, however, there is no full $GL(2,\mathbb{R})$ invariance, especially no invariance under the special projective transformation, as we have mentioned

previously in Section 3. It turns out that the dilatation and translation invariance are insufficient to extract the general form of the type A′ gauged Hamiltonians, namely, the operators which can be transformed to Schrödinger operators from the most general type A′ quasi-solvable second-order operators (4.14).

Let us first review the general condition under which a second-order linear differential operator of several variables can be cast into a Schrödinger operator [23]. Suppose the operator under consideration $\tilde{\mathcal{H}}$ has the following form in variables z:

$$\tilde{\mathcal{H}} = -\sum_{i,j=1}^{M} P_{ij}(z)\frac{\partial^2}{\partial z_i \partial z_j} + \sum_{i=1}^{M} S_i(z)\frac{\partial}{\partial z_i} - T(z). \tag{5.1}$$

Then, it is readily shown that $\tilde{\mathcal{H}}$ can be cast into a Schrödinger operator by a gauge transformation and a change of variables $z_i = z(q_i)$ such that

$$\mathrm{e}^{-\mathcal{W}}\tilde{\mathcal{H}}\mathrm{e}^{\mathcal{W}} = -\frac{1}{2}\sum_{i=1}^{M}\frac{\partial^2}{\partial q_i^2} + V(q), \tag{5.2}$$

if and only if the following conditions are satisfied:

$$P_{ij}(z) + P_{ji}(z) = 0 \quad (i > j), \tag{5.3}$$

$$(z_i')^2 = 2P_{ii}(z), \tag{5.4}$$

$$\frac{\partial \mathcal{W}}{\partial q_i} = \frac{S_i(z)}{z_i'} + \frac{z_i''}{2z_i'}, \tag{5.5}$$

where z_i' denotes the derivative of z_i with respect to q_i. The first condition (5.3) in general consists of a set of algebraic identities. The second and third conditions (5.4)–(5.5) on the other hand are sets of differential equations and do not necessarily have a solution. In order that the second condition has a solution, the r.h.s. of Eq. (5.4) must depend only on the single variable z_i:

$$(z_i')^2 = 2P_{ii}(z) = 2A(z_i), \tag{5.6}$$

since we have assumed that the change of variables is determined by a single function of a single variable $z_i = z(q_i)$ and thus the l.h.s. of Eq. (5.4) depends solely on the variable q_i. Furthermore, in order that the third condition (5.5) has a solution, the following integrability condition must be fulfilled for all $i \neq l$:

$$\frac{\partial}{\partial q_l}\frac{\partial \mathcal{W}}{\partial q_i} = \frac{\partial}{\partial q_i}\frac{\partial \mathcal{W}}{\partial q_l} \quad \Leftrightarrow \quad \frac{1}{A(z_i)}\frac{\partial S_i(z)}{\partial z_l} = \frac{1}{A(z_l)}\frac{\partial S_l(z)}{\partial z_i}, \tag{5.7}$$

where Eq. (5.6) is employed. Therefore, the operator $\tilde{\mathcal{H}}$ can be transformed

to a Schrödinger operator if it has the form of

$$\tilde{\mathcal{H}} = -\sum_{i=1}^{M} A(z_i)\frac{\partial^2}{\partial z_i^2} + \sum_{i=1}^{M} S_i(z)\frac{\partial}{\partial z_i} - T(z), \tag{5.8}$$

with $A(z_i)$ and $S_i(z)$ satisfying Eqs. (5.6) and (5.7).

Next, we note that in our present case we have made the additional change of variables from z to σ, Eq. (3.5). From the formula (A.4), the part of the second-order operators in Eq. (4.14) has the following form in terms of z:

$$-\sum_{k,l=1}^{M}\Big[\boldsymbol{A}_0(\sigma)kl\sigma_k\sigma_l + \boldsymbol{A}_{kl}(\sigma)\Big]\frac{\partial^2}{\partial\sigma_k\partial\sigma_l} = -\sum_{i,j=1}^{M} P_{ij}(z)\frac{\partial^2}{\partial z_i\partial z_j}, \tag{5.9}$$

with

$$P_{ij}(z) = \frac{\sum_{k,l=1}^{M}(-1)^{k+l}z_i^{M-k}z_j^{M-l}\Big[\boldsymbol{A}_0(\sigma)kl\sigma_k\sigma_l + \boldsymbol{A}_{kl}(\sigma)\Big]}{\prod_{m(\neq i)}^{M}(z_i - z_m)\prod_{n(\neq j)}^{M}(z_j - z_n)}. \tag{5.10}$$

To satisfy the condition (5.6), it is evident that the denominator in the r.h.s. of Eq. (5.10) for $P_{ii}(z)$ must be completely canceled with a part of the numerator; otherwise, $P_{ii}(z)$ cannot be a function of the single variable z_i. As a consequence, the part of the second-order operators in Eq. (4.14) which satisfy the conditions (5.3) and (5.6) must have the following form:

$$-\sum_{k,l=1}^{M}\Big[\boldsymbol{A}_0(\sigma)kl\sigma_k\sigma_l + \boldsymbol{A}_{kl}(\sigma)\Big]\frac{\partial^2}{\partial\sigma_k\partial\sigma_l} = -\sum_{i=1}^{M}\left(\sum_{p=0}^{n} a_p z_i^p\right)\frac{\partial^2}{\partial z_i^2}, \tag{5.11}$$

where a_p are constants. Then, the next task is to examine which of a_p can be non-zero free parameters. From a *dimensional analysis*, we easily see that the each term in the r.h.s. of Eq. (5.11) for a fixed p must be expressed in terms of σ as

$$\sum_{i=1}^{M} z_i^p\frac{\partial^2}{\partial z_i^2} = \sum_{k,l=1}^{M}\sum_{\{\overline{m}_i\}_{k+l+p-2}} A^{[p]}_{\{\overline{m}_i\}_{k+l+p-2},kl}\prod_{i=1}^{M}\sigma_i^{\overline{m}_i}\frac{\partial^2}{\partial\sigma_k\partial\sigma_l}, \tag{5.12}$$

with a suitable set of constants $A^{[p]}_{\{\overline{m}_i\}_{k+l+p-2},kl}$. On the other hand, we can read from Eqs. (4.15a)–(4.15b) that the l.h.s. of Eq. (5.11) contains operators of the form

$$\prod_{i=1}^{M}\sigma_i^{m_i}\frac{\partial^2}{\partial\sigma_k\partial\sigma_l}, \tag{5.13}$$

only for $0 \leq \sum_{i=1}^{M} im_i \leq k+l+2$. From this fact and Eq. (5.12), we readily know that Eq. (5.11) cannot be satisfied unless $a_p = 0$ for $p > 4$. From Eq. (4.15b), we see that the l.h.s. of Eq. (5.11) contains *all* the set of operators of the form

(5.13) for $(0 \leq)k+l-2 \leq \sum_{i=1}^{M} im_i \leq k+l$, and hence Eq. (5.11) can be satisfied for all non-zero values of a_0, a_1, and a_2, with suitable values of the parameters $A_{\{m_i\}_{k+l},kl}$ in $\boldsymbol{A}_{kl}(\sigma)$. However, only the second term in the r.h.s. of Eq. (4.15b) produces the operators of the form (5.13) for $\sum_{i=1}^{M} im_i = k+l+1$ and is insufficient to express the operator in the r.h.s. of Eq. (5.11) for $p=3$, cf. Section B.5 in Ref. [11]. Hence, Eq. (5.11) cannot be satisfied unless $a_3 = 0$ and $A_{10,\{\overline{m}_i\}_k,k} = 0$. A similar observation for $\sum_{i=1}^{M} = k+l+2$ leads to $a_4 = 0$ and $A_{10,10} = A_{20,00} = 0$. Summarizing the analyses so far, we conclude that $a_p = 0$ for all $p > 2$ and thus the coefficient of the second-order operator of the type A′ gauged Hamiltonians of the form (5.8) must be

$$A(z_i) = a_2 z_i^2 + a_1 z_i + a_0. \tag{5.14}$$

Simultaneously, all the following parameters in $\boldsymbol{A}_0(\sigma)$ and $\boldsymbol{A}_{kl}(\sigma)$ must vanish:

$$A_{10,\{\overline{m}_i\}_k,k} = A_{10,10} = A_{20,00} = 0. \tag{5.15}$$

Next, we shall examine the part of the first-order operators. With the aid of Eq. (A.4), the part of the first-order differential operators in the general type A′ operators (4.14) reads

$$\sum_{k=1}^{M} \Big[\boldsymbol{B}_0(\sigma) k\sigma_k - \boldsymbol{B}_k(\sigma) \Big] \frac{\partial}{\partial \sigma_k} = \sum_{i=1}^{M} S_i(z) \frac{\partial}{\partial z_i}, \tag{5.16}$$

with

$$S_i(z) = \frac{\sum_{k=1}^{M} (-1)^{k+1} z_i^{M-k} \Big[\boldsymbol{B}_0(\sigma) k\sigma_k - \boldsymbol{B}_k(\sigma) \Big]}{\prod_{j(\neq i)}^{M} (z_i - z_j)} \equiv \frac{\mathcal{S}_i(z)}{\prod_{j(\neq i)}^{M} (z_i - z_j)}. \tag{5.17}$$

Thus, the derivative of $S_i(z)$ with respect to $z_l (l \neq i)$ in our case is calculated as

$$\frac{\partial S_i(z)}{\partial z_l} = \frac{(z_i - z_l)\partial_l \mathcal{S}_i(z) + \mathcal{S}_i(z)}{(z_i - z_l)^2 \prod_{j(\neq i,l)}^{M} (z_i - z_j)}. \tag{5.18}$$

Therefore, the integrability condition (5.7) can be satisfied if and only if the numerator of the r.h.s. of Eq. (5.18) has the following form:

$$(z_i - z_l)\partial_l \mathcal{S}_i(z) + \mathcal{S}_i(z) = f_1(z_i, z_l) f_2(\sigma) A(z_i) \prod_{j(\neq i,l)}^{M} (z_i - z_j), \tag{5.19}$$

where $f_1(z_i, z_l) = f_1(z_l, z_i)$ and $f_2(\sigma)$ is a function depending solely on the elementary symmetric polynomials.

On the other hand, under the condition (5.15) satisfied, the coefficient of the

first-order differential operators in Eq. (4.14) reads

$$\boldsymbol{B}_0(\sigma)k\sigma_k - \boldsymbol{B}_k(\sigma) = -B_{10}k\sigma_1\sigma_k + \sum_{\{m_i\}_k} B_{\{m_i\}_k,k}\prod_{i=1}^{M}\sigma_i^{m_i}. \tag{5.20}$$

Hence, $\mathcal{S}_i(z)$ defined by Eq. (5.17) is a polynomial of degree at most $M+1$ in the variables z. From Eq. (5.19), we conclude that the combination of the functions $f_1(z_i, z_l)f_2(\sigma)A(z_i)$ in the r.h.s. of Eq. (5.19) must be a polynomial of degree at most 3 in the variables z. Let us first examine the highest-degree term. It comes from the first term in the r.h.s. of Eq. (5.20). The corresponding term in $\mathcal{S}_i(z)$ is

$$\mathcal{S}_i(z) = -B_{10}\sigma_1\sum_{k=1}^{M}(-1)^{k+1}kz_i^{M-k}\sigma_k. \tag{5.21}$$

With the aid of the formulas (A.1)–(A.3), we have

$$(z_i - z_l)\partial_l\mathcal{S}_i(z) + \mathcal{S}_i(z) = -B_{10}z_i(z_i - z_l)^2\prod_{j(\neq i,l)}^{M}(z_i - z_j). \tag{5.22}$$

Comparing with Eq. (5.19), we see that $f_1(z_i, z_l) = (z_i - z_l)^2$ and $f_2(\sigma) =$ const. for the highest-degree term. However, the remaining term $-B_{10}z_i$ is a monomial of first-degree. This means that the highest-degree term (together with lower-degree terms) cannot be expressed as Eq. (5.19) unless $A(z_i)$ is a polynomial of first-degree. In other words, the highest-degree term can exist if and only if $a_2 = 0$ in Eq. (5.14). Hence, the possible form of the formula which includes the highest-degree term must be

$$(z_i - z_l)\partial_l\mathcal{S}_i(z) + \mathcal{S}_i(z) = -2gA(z_i)(z_i - z_l)^2\delta_{a_2,0}\prod_{j(\neq i,l)}^{M}(z_i - z_j), \tag{5.23}$$

where g is a constant. It is evident that the term proportional to a_1 in the r.h.s. of Eq. (5.23) comes from the highest-degree term given by Eq. (5.22) and thus the constant B_{10} in Eq. (5.22) is expressed as

$$B_{10} = 2ga_1\delta_{a_2,0}. \tag{5.24}$$

We will later see that there in fact exists the term in $S_i(z)$ which corresponds to the term proportional to a_0 in the r.h.s. of Eq.(5.23). Next, we shall examine the lower-degree terms, namely, the terms of degree less than $M+1$ in $\mathcal{S}_i(z)$ which result in the terms of degree less than 3 in the combination $f_1(z_i, z_l)f_2(\sigma)A(z_i)$. As we have already obtained the terms which can only exist in the case of $a_2 = 0$, we can assume here that $a_2 \neq 0$ and thus $A(z_i)$ is a strictly second-degree polynomial. Hence, only the possible form for the

lower-degree terms is $f_1(z_i, z_l) f_2(\sigma) = -2c$, c is a constant, and we have

$$(z_i - z_l)\partial_l \mathcal{S}_i(z) + \mathcal{S}_i(z) = -2cA(z_i) \prod_{j(\neq i,l)}^{M} (z_i - z_j). \tag{5.25}$$

Substituting the higher-degree term (5.23) together with the lower-degree term (5.25) in Eq. (5.18), we eventually obtain

$$\frac{\partial S_i(z)}{\partial z_l} = -2gA(z_i)\delta_{a_2,0} - 2c\frac{A(z_i)}{(z_i - z_l)^2} \quad (i \neq l). \tag{5.26}$$

This set of differential equations can be easily integrated as

$$S_i(z) = -Q(z_i) - 2g\sigma_1 A(z_i)\delta_{a_2,0} - 2c \sum_{j(\neq i)}^{M} \frac{A(z_i)}{z_i - z_j}, \tag{5.27}$$

where $Q(z_i)$ is a function of a single variable. This term should come from at most Mth-degree terms in $\mathcal{S}_i(z)$ which cancel the denominator of the last term in Eq. (5.17) so that it depends solely on the single variable z_i. The denominator is of degree $M-1$ and thus $Q(z_i)$ must be a polynomial of at most first-degree:

$$Q(z_i) = b_1 z_i + b_0. \tag{5.28}$$

Next, we shall check whether Eq. (5.16) actually holds for Eqs. (5.20) and (5.27). From the formulas (A.5), the r.h.s. of Eq. (5.16) with $S_i(z)$ given by Eq. (5.27) is expressed as

$$\begin{aligned}\sum_{i=1}^{M} S_i(z)\frac{\partial}{\partial z_i} = \sum_{k=1}^{M}\Big\{&(M-k+1)(M-k+2)ca_0\sigma_{k-2} \\ &- (M-k+1)\Big[b_0 + (M-k)ca_1\Big]\sigma_{k-1} - k\Big[b_1 + (2M-k-1)ca_2\Big]\sigma_k \\ &- 2(M-k+1)ga_0\delta_{a_2,0}\sigma_1\sigma_{k-1} - 2a_1 g\delta_{a_2,0}k\sigma_1\sigma_k\Big\}\frac{\partial}{\partial \sigma_k}.\end{aligned} \tag{5.29}$$

We now easily see that all the terms in the braces in the r.h.s. of the above equation are contained in Eq. (5.20) for each fixed k. Hence, all the term in Eq. (5.27) can indeed exist. Substituting Eq. (5.27) in Eq. (5.5) and using Eq. (5.6), we have

$$\frac{\partial \mathcal{W}}{\partial z_i} = -\frac{Q(z_i)}{2A(z_i)} + \frac{A'(z_i)}{4A(z_i)} - g\sigma_1\delta_{a_2,0} - c\sum_{j(\neq i)}^{M} \frac{1}{z_i - z_j}. \tag{5.30}$$

Again, we can easily integrate the above set of differential equations to obtain

$$\begin{aligned}\mathcal{W} = &- \sum_{i=1}^{M} \int \mathrm{d}z_i \, \frac{Q(z_i)}{2A(z_i)} + \frac{1}{4} \sum_{i=1}^{M} \ln \bigl| A(z_i) \bigr| - \frac{g}{2} \sigma_1^2 \delta_{a_2,0} \\ &- c \sum_{i<j}^{M} \ln |z_i - z_j|,\end{aligned} \tag{5.31}$$

where we have omitted the integral constant.

Finally from Eqs. (5.8) and (5.27), we find that the gauged Hamiltonians $\tilde{H}_{\mathcal{N}}^{(A')}$ which preserve the type A′ space (3.8) must have the following form:

$$\tilde{H}_{\mathcal{N}}^{(A')} = - \sum_{i=1}^{M} A(z_i) \frac{\partial^2}{\partial z_i^2} - \sum_{i=1}^{M} B_i(z) \frac{\partial}{\partial z_i} - 2c \sum_{i \neq j}^{M} \frac{A(z_i)}{z_i - z_j} \frac{\partial}{\partial z_i} - \boldsymbol{C}(\sigma), \tag{5.32}$$

where $A(z_i)$ and $B_i(z)$ are given by Eq. (5.14) and by

$$B_i(z) = 2g\sigma_1 A(z_i) \delta_{a_2,0} + Q(z_i), \tag{5.33}$$

respectively. From Eqs. (4.15e), (5.15), and (5.24), the function $\boldsymbol{C}$ in Eq. (5.32) is calculated as

$$\boldsymbol{C}(\sigma) = -2(\mathcal{N} - 1) g a_1 \sigma_1 \delta_{a_2,0} + c_0. \tag{5.34}$$

It is easily shown that the gauged Hamiltonian (5.32) can be actually cast into a Schrödinger operator by a gauge transformation

$$\begin{aligned}H_{\mathcal{N}}^{(A')} &= \mathrm{e}^{-\mathcal{W}} \tilde{H}_{\mathcal{N}}^{(A')} \mathrm{e}^{\mathcal{W}} \\ &= -\frac{1}{2} \sum_{i=1}^{M} \frac{\partial^2}{\partial q_i^2} + \frac{1}{2} \sum_{i=1}^{M} \left[\left(\frac{\partial \mathcal{W}}{\partial q_i} \right)^2 - \frac{\partial^2 \mathcal{W}}{\partial q_i^2} \right] - \boldsymbol{C}(\sigma),\end{aligned} \tag{5.35}$$

if the gauge potential $\mathcal{W}$ is chosen as Eq. (5.31) and the function $z(q)$ which determines the change of variables satisfies

$$z'(q)^2 = 2A\bigl(z(q)\bigr) = 2\bigl(a_2 z(q)^2 + a_1 z(q) + a_0\bigr). \tag{5.36}$$

In this case, Eqs. (4.16), (5.15), and (5.24) tell us that the gauged Hamiltonian (5.32) is not only quasi-solvable but also solvable if and only if

$$B_{10} = g a_1 \delta_{a_2,0} = 0. \tag{5.37}$$

It is apparent from the construction that the Hamiltonian (5.35) preserves the space $\mathcal{V}_{\mathcal{N};M}^{(A')}$ defined by

$$\mathcal{V}_{\mathcal{N};M}^{(A')} = \mathrm{e}^{-\mathcal{W}} \, \tilde{\mathcal{V}}_{\mathcal{N};M}^{(A')}. \tag{5.38}$$

Hence, the type A′ Hamiltonian $H_{\mathcal{N}}^{(A')}$ can be (locally) diagonalized in the finite dimensional space (5.38). We shall thus call the space (5.38) the *solvable sector* of $H_{\mathcal{N}}^{(A')}$.

6 Classification of the Models

We shall now explicitly compute the concrete form of the type A′ quantum Hamiltonians. From Eqs. (5.14), (5.30), and (5.34), the potential term in Eq. (5.35) is explicitly calculated in terms of z as

$$\begin{aligned} V = {} & \sum_{i=1}^{M} \frac{1}{16A(z_i)} \Big[2Q(z_i) - A'(z_i)\Big]\Big[2Q(z_i) - 3A'(z_i)\Big] \\ & + g\left[\sigma_1(z)\sum_{i=1}^{M} Q(z_i) + \Big(g\sigma_1(z)^2 + 1\Big)\sum_{i=1}^{M} A(z_i) + a_1(M,\mathcal{N})\sum_{i=1}^{M} z_i\right]\delta_{a_2,0} \\ & + c(c-1)\sum_{i<j}^{M} \frac{A(z_i) + A(z_j)}{(z_i - z_j)^2} + V_0, \end{aligned} \tag{6.1}$$

where, and in what follows, V_0 denotes an arbitrary constant, and the coupling constant $a_1(M,\mathcal{N})$ is given by,

$$a_1(M,\mathcal{N}) = \Big[2(\mathcal{N}-1) + M(M-1)c\Big]a_1. \tag{6.2}$$

From Eq. (5.36), the change of variable is determined by the following integral:

$$\pm(q - q_0) = \int \frac{\mathrm{d}z}{\sqrt{2(a_2 z^2 + a_2 z + a_1)}}. \tag{6.3}$$

In contrast to the type A case where the systems are constructed from the $\mathfrak{sl}(M+1)$ generators, our present models do not have full $GL(2,\mathbb{R})$ invariance as has been mentioned previously in Section 3. However, the remaining invariance under the dilatation and translation enables us to classify the type A′ models. Indeed, it is readily shown that $A(z)$ can be cast into one of the canonical forms listed in Table 1 by a combination of the dilatation and translation.

Furthermore, we note that from Eq. (5.36), a rescaling of the coefficients a_i, b_i, c_0 by an overall constant factor ν has the following effect on the change of variable $z(q)$:

$$z(q\,;\nu a_i, \nu b_i, \nu c_0) = z(\sqrt{\nu}\,q\,; a_i, b_i, c_0). \tag{6.4}$$

From this equation and Eqs. (5.14), (5.28), (5.31), and (6.1), we easily obtain

Table 1
Canonical forms for the polynomial $A(z)$, (5.14), where ν is a positive real constant.

Case	Canonical Form
I	$1/2$
II	$2z$
III	$\pm 2\nu z^2$
IV	$\pm 2\nu(z^2 - 1)$
IV$'$	$\pm 2\nu(z^2 + 1)$

the identities

$$\mathcal{W}(q\,;\nu a_i,\nu b_i,\nu c_0) = \mathcal{W}(\sqrt{\nu}\,q\,;a_i,b_i,c_0), \tag{6.5a}$$
$$V(q\,;\nu a_i,\nu b_i,\nu c_0) = \nu\,V(\sqrt{\nu}\,q\,;a_i,b_i,c_0). \tag{6.5b}$$

We shall therefore set $\nu = 1$ in the canonical forms in Cases II–IV$^{(')}$, the models corresponding to an arbitrary value of ν following easily from Eqs. (6.4) and (6.5). It should also be obvious from Eq. (6.3) that the change of variable $z(q)$, and hence the potential V determining each model, are defined up to the transformation $q \mapsto \pm(q - q_0)$, where $q_0 \in \mathbb{R}$ is a constant. The solvability condition (5.37) implies that except for the model with $g \neq 0$ corresponding to Case II in Table 1 all the obtainable models are not only quasi-solvable but also solvable.

As we will see below, the potentials in Case I, III, and IV$'$ have singularities at $q_i = q_j$ for all $i \neq j$ in the subspace $\{-\Omega \le q_i < \Omega\} \in \mathbb{R}^M$ where Ω is a submultiple of the real period of the potential or $\Omega = \infty$ when the potential is non-periodic. Similarly, the potentials in Case II and IV are singular at $q_i = 0$ and $q_i = \pm q_j$ for all $i \neq j$ in the same subspace of $\mathbb{R}^M$. Hence, the Hamiltonians are naturally defined on

$$0 < q_M < \cdots < q_1 < \Omega. \tag{6.6}$$

Normalizability of the solvable sector (5.38) on the space (6.6) depends on the behavior of the gauge potential $\mathcal{W}(q)$ in the each case. The finiteness of the L^2 norm of the two-body wave function in the solvable sector $\mathcal{V}^{(A')}_{\mathcal{N};M}$ in general leads $c > -1/2$, where c denotes the coupling constant of the two-body interaction appeared in the last line of Eq. (6.1).

6.1 *Case I: $A(z) = 1/2$*

Change of variable: $z(q) = q$.

Potential:

$$
\begin{aligned}
V(q) = {} & g\left(\frac{M}{2}g + b_1\right)\left(\sum_{i=1}^{M} q_i\right)^2 + Mb_0 g\sum_{i=1}^{M} q_i + \frac{1}{2}\sum_{i=1}^{M}(b_1 q_i + b_0)^2 \\
& + c(c-1)\sum_{i<j}^{M}\frac{1}{(q_i - q_j)^2} + V_0.
\end{aligned} \tag{6.7}
$$

Gauge potential:

$$
\mathcal{W}(q) = -\frac{g}{2}\left(\sum_{i=1}^{M} q_i\right)^2 - \frac{b_1}{2}\sum_{i=1}^{M} q_i^2 - b_0\sum_{i=1}^{M} q_i - c\sum_{i<j}^{M}\ln|q_i - q_j|. \tag{6.8}
$$

When $g = 0$, this case corresponds to the rational A_{M-1} type Calogero–Sutherland model [5] and is identical with the solvable model corresponding to Case I of the type A models, Eqs. (7.8)–(7.9) with $b_2 = 0$ in Ref. [11]. Hence, the above model provides an example of deformed Calogero–Sutherland models which preserve quantum solvability. The parameter b_0 corresponds to the translational degree of freedom and is irrelevant. In fact, if we employ the translational freedom mentioned earlier below Eqs. (6.5) and apply the translation $q_i \mapsto q_i - b_0/(b_1 + Mg)$ in Eqs. (6.7)–(6.8), the potential and gauge potential become

$$
V(q) = g\left(\frac{M}{2}g + b_1\right)\left(\sum_{i=1}^{M} q_i\right)^2 + \frac{b_1^2}{2}\sum_{i=1}^{M} q_i^2 + c(c-1)\sum_{i<j}^{M}\frac{1}{(q_i - q_j)^2} + V_0, \tag{6.9}
$$

$$
\mathcal{W}(q) = -\frac{g}{2}\left(\sum_{i=1}^{M} q_i\right)^2 - \frac{b_1}{2}\sum_{i=1}^{M} q_i^2 - c\sum_{i<j}^{M}\ln|q_i - q_j|, \tag{6.10}
$$

and have no dependence on b_0 any more. It is now obvious that the model is exactly the rational A_{M-1} type Calogero–Sutherland model with the center-of-mass coordinate subjected to the harmonic oscillator potential. Since we can easily separate the center-of-mass coordinate from the others, quantum solvability of the above model is readily understood.

The one-body part of the potential has no singularities and hence a natural choice is $\Omega = \infty$. In this choice, the form of the gauge potential (6.8) tells us that the solvable sector (5.38) is square integrable on the space (6.6) as long as $g < 0$, $b_1 < 0$, and $c > -1/2$. Hence, the model (6.7) is exactly solvable on Eq. (6.6) in these parameter regions.

6.2 Case II: $A(z) = 2z$

Change of variable: $z(q) = q^2$.

Potential:

$$\begin{aligned} V(q) = {} & \left(\frac{b_1^2}{8} + 2(2\mathcal{N} - 1)g + M b_0 g + 2M(M-1)cg \right) \sum_{i=1}^{M} q_i^2 \\ & + 2g^2 \left(\sum_{i=1}^{M} q_i^2 \right)^3 + b_1 g \left(\sum_{i=1}^{M} q_i^2 \right)^2 + \frac{(b_0 - 1)(b_0 - 3)}{8} \sum_{i=1}^{M} \frac{1}{q_i^2} \\ & + c(c-1) \sum_{i<j}^{M} \left[\frac{1}{(q_i - q_j)^2} + \frac{1}{(q_i + q_j)^2} \right] + V_0. \end{aligned} \tag{6.11}$$

Gauge potential:

$$\mathcal{W}(q) = -\frac{g}{2} \left(\sum_{i=1}^{M} q_i^2 \right)^2 - \frac{b_1}{4} \sum_{i=1}^{M} q_i^2 - \frac{b_0 - 1}{2} \sum_{i=1}^{M} \ln |q_i| - c \sum_{i<j}^{M} \ln \left| q_i^2 - q_j^2 \right|. \tag{6.12}$$

This case corresponds to the rational BC_M type Calogero–Sutherland model [24] when $g = 0$. Furthermore, the above model is quasi-solvable when $g \neq 0$. To the best of our knowledge, it is new and provides another quasi-solvable deformation of Calogero–Sutherland models which is different from the Inozemtsev type classified in Ref. [11]. In this respect, we should refer to similar quasi-solvable models in the literature, namely, Eq. (23) in Ref. [7] and Eq. (3.7) in Ref. [25]. These models are deformations of the rational A_{M-1} type Calogero–Sutherland system and thus their two-body interaction terms are different from our BC_M type. Furthermore, the solvable sectors of these models are spanned by monomials of a *single* variable while that of our present model is by monomials of M variables (3.8).

The one-body part of the potential is only singular at $q_i = 0$ and hence a natural choice is $\Omega = \infty$. In this choice, we see from Eq. (6.12) that the solvable sector (5.38) is square integrable on the space (6.6) as long as $g < 0$ and $c > -1/2$. Hence, the model (6.11) is quasi-exactly solvable on the space (6.6) in these parameter regions. When $M = 1$, the above model becomes

$$V(q) = \frac{1}{8} q^2 (4gq^2 + b_1)^2 + (4\mathcal{N} + b_0 - 2) g q^2 + \frac{(b_0 - 1)(b_0 - 3)}{8q^2} + V_0, \tag{6.13}$$

and thus exactly reduces to the well-known one-body quasi-solvable sextic anharmonic oscillator, classified in Case II of the type A models, Eqs. (7.11)–(7.12) with $b_2 = 4g$ and $c_1 = (b_0 - 1)/2$ in Ref. [11].

6.3 Case IIIa: $A(z) = 2z^2$

Change of variable: $z(q) = \mathrm{e}^{2q}$.

Potential:

$$V(q) = \frac{b_0(b_1-4)}{4}\sum_{i=1}^{M}\mathrm{e}^{-2q_i} + \frac{b_0^2}{8}\sum_{i=1}^{M}\mathrm{e}^{-4q_i} + c(c-1)\sum_{i<j}^{M}\frac{1}{\sinh^2(q_i-q_j)} + V_0. \tag{6.14}$$

Gauge potential:

$$\begin{aligned}\mathcal{W}(q) = {}& \frac{b_0}{4}\sum_{i=1}^{M}\mathrm{e}^{-2q_i} - \left(\frac{b_1}{2} - 1 + (M-1)c\right)\sum_{i=1}^{M} q_i \\ & - c\sum_{i<j}^{M}\ln\left|\sinh(q_i-q_j)\right|.\end{aligned} \tag{6.15}$$

This model is the hyperbolic A_{M-1} Calogero–Sutherland model [6] in the external Morse potential and completely the same as the model corresponding to Case III of the type A models, Eqs. (7.15)–(7.16) with $b_2 = 0$ in Ref. [11].

6.4 Case IIIb: $A(z) = -2z^2$

The formulas of the potential and gauge potential for this case can be easily reduced from Eqs. (6.14)–(6.15) using Eqs. (6.5) with $\nu = -1$. The change of variable is $z(q) = \mathrm{e}^{2\mathrm{i}q}$.

6.5 Case IVa: $A(z) = 2(z^2-1)$

Change of variable: $z(q) = \cosh 2q$.

Potential:

$$\begin{aligned}V(q) = {}& \frac{(b_1-b_0-2)(b_1-b_0-6)}{8}\sum_{i=1}^{M}\frac{1}{\sinh^2 2q_i} + \frac{b_0(b_1-4)}{8}\sum_{i=1}^{M}\frac{1}{\sinh^2 q_i} \\ & + c(c-1)\sum_{i<j}^{M}\left[\frac{1}{\sinh^2(q_i-q_j)} + \frac{1}{\sinh^2(q_i+q_j)}\right] + V_0.\end{aligned} \tag{6.16}$$

Gauge potential:

$$\mathcal{W}(q) = -\frac{b_1 - 2}{4}\sum_{i=1}^{M} \ln|\sinh 2q_i| - \frac{b_0}{4}\sum_{i=1}^{M} \ln|\tanh q_i| \\ - c\sum_{i<j}^{M} \ln\Big|\sinh(q_i - q_j)\sinh(q_i + q_j)\Big|. \tag{6.17}$$

This model is the hyperbolic BC_M Calogero–Sutherland model [24] and completely the same as the model corresponding to Case IV of the type A models, Eqs. (7.22)–(7.23) with $b_2 = 0$ in Ref. [11].

6.6 Case IVb: $A(z) = -2(z^2 - 1)$

The formulas of the potential and gauge potential for this case can be easily reduced from Eqs. (6.16)–(6.17) using Eqs. (6.5) with $\nu = -1$. The change of variable is $z(q) = \cos 2q$.

6.7 Case IV′a: $A(z) = 2(z^2 + 1)$

Change of variable: $z = \sinh 2q$.

Potential:

$$V(q) = \frac{b_0^2 - (b_1 - 2)(b_1 - 6)}{8}\sum_{i=1}^{M} \frac{1}{\cosh^2 2q_i} + \frac{b_0(b_1 - 4)}{4}\sum_{i=1}^{M} \frac{\sinh 2q_i}{\cosh^2 2q_i} \\ + c(c-1)\sum_{i<j}^{M}\left[\frac{1}{\sinh^2(q_i - q_j)} - \frac{1}{\cosh^2(q_i + q_j)}\right] + V_0. \tag{6.18}$$

Gauge potential:

$$\mathcal{W}(q) = -\frac{b_1 - 2}{4}\sum_{i=1}^{M} \ln|\cosh 2q_i| - \frac{b_0}{4}\sum_{i=1}^{M} \operatorname{gd} 2q_i \\ - c\sum_{i<j}^{M} \Big|\sinh(q_i - q_j)\cosh(q_i + q_j)\Big|, \tag{6.19}$$

where $\operatorname{gd} q = \arctan(\sinh q)$ is the Gudermann function. This model is another hyperbolic BC_M Calogero–Sutherland model and completely the same as the model corresponding to Case IV′ of the type A models, Eqs. (7.26)–(7.27) with $b_2 = 0$ in Ref. [11].

6.8 *Case IV′b:* $A(z) = -2(z^2 + 1)$

The formulas of the potential and gauge potential for this case can be easily reduced from Eqs. (6.18)–(6.19) using Eqs. (6.5) with $\nu = -1$. The change of variable is $z(q) = \mathrm{i}\sin 2q$.

7 A New Type C Generalization Based on the Type A′ Scheme

In the preceding sections, we have investigated (at most) second-order linear differential operators preserving the type A′ space (3.8). This new generalization of the single-variable type A space to several variables suggests a new multivariate generalization of the single-variable type C space. The type C monomial space of a single variable z is defined by [17]

$$\tilde{\mathcal{V}}^{(C)}_{\mathcal{N}_1,\mathcal{N}_2} = \tilde{\mathcal{V}}^{(A)}_{\mathcal{N}_1} \oplus z^{\lambda}\, \tilde{\mathcal{V}}^{(A)}_{\mathcal{N}_2}, \tag{7.1}$$

where $\tilde{\mathcal{V}}^{(A)}_{\mathcal{N}_i}$ $(i = 1, 2)$ is a type A monomial space of dimension $\mathcal{N}_i$ defined by Eq. (3.6), $\mathcal{N}_1$ and $\mathcal{N}_2$ are positive integers satisfying $\mathcal{N}_1 \geq \mathcal{N}_2$, and λ is a real number with the restriction

$$\lambda \in \mathbb{R} \setminus \{-\mathcal{N}_2, -\mathcal{N}_2 + 1, \dots, \mathcal{N}_1\}, \tag{7.2}$$

and with $\lambda \neq -\mathcal{N}_2 - 1, \mathcal{N}_1 + 1$ if $\mathcal{N}_1 = 1$ or $\mathcal{N}_2 = 1$. In the previous paper [18], we generalized the type C space of a single variable (7.1) to several variables as follows:

$$\tilde{\mathcal{V}}^{(C)}_{\mathcal{N}_1,\mathcal{N}_2;M} = \tilde{\mathcal{V}}^{(A)}_{\mathcal{N}_1;M} \oplus \sigma_M^{\lambda}\, \tilde{\mathcal{V}}^{(A)}_{\mathcal{N}_2;M}, \tag{7.3}$$

where $\tilde{\mathcal{V}}^{(A)}_{\mathcal{N}_i;M}$ $(i = 1, 2)$ is a multivariate type A space defined by Eq. (3.7). The above generalization scheme based on the type A space (3.7) now strongly suggests another generalization based on the type A′ space (3.8) as the following:

$$\tilde{\mathcal{V}}^{(C')}_{\mathcal{N}_1,\mathcal{N}_2;M} = \tilde{\mathcal{V}}^{(A')}_{\mathcal{N}_1;M} \oplus \sigma_M^{\lambda}\, \tilde{\mathcal{V}}^{(A')}_{\mathcal{N}_2;M}. \tag{7.4}$$

Indeed, the latter space also reduces to the single-variable type C space (7.1) when $M = 1$. Hence, Eq. (7.4) would provide a new multivariate generalization of the type C space and we hereafter call the space (7.4) *type C′*. In this section, we shall investigate (at most) second-order linear differential operators which preserve the type C′ space (7.4).

First of all, the set of linearly independent first-order differential operators

preserving the type C′ space is given by,

$$F_{\{m_i\}_{\bar{k}},\bar{k}} = \prod_{i=1}^{M} \sigma_i^{m_i} \frac{\partial}{\partial \sigma_{\bar{k}}} \quad (\bar{k} = 1, \dots, M-1), \tag{7.5}$$

$$E_{MM} \equiv \sigma_M \frac{\partial}{\partial \sigma_M}. \tag{7.6}$$

Similarly, the set of the linearly independent second-order differential operators preserving the type C′ space is as follows:

$$F_{\{m_i\}_{\bar{k}+\bar{l}},\bar{k}\bar{l}} = \prod_{i=1}^{M} \sigma_i^{m_i} \frac{\partial^2}{\partial \sigma_{\bar{k}} \partial \sigma_{\bar{l}}} \quad (\bar{k}, \bar{l} = 1, \dots, M-1;\ \bar{k} \geq \bar{l}), \tag{7.7}$$

$$F_{\{m_i\}_{\bar{k}},\bar{k}} E_{MM} = \prod_{i=1}^{M} \sigma_i^{m_i} \sigma_M \frac{\partial^2}{\partial \sigma_M \partial \sigma_{\bar{k}}} \quad (\bar{k} = 1, \dots, M-1), \tag{7.8}$$

$$F_{\{m_i\}_M,M}(E_{MM} - \lambda) = \prod_{i=1}^{M} \sigma_i^{m_i} \frac{\partial}{\partial \sigma_M} \left(\sigma_M \frac{\partial}{\partial \sigma_M} - \lambda \right), \tag{7.9}$$

$$F_{10,00} \equiv \sigma_1 \left(\mathcal{N}_1 - 1 - \sum_{k=1}^{M} k \sigma_k \frac{\partial}{\partial \sigma_k} \right) \left(M\lambda + \mathcal{N}_2 - 1 - \sum_{l=1}^{M} l \sigma_l \frac{\partial}{\partial \sigma_l} \right). \tag{7.10}$$

Therefore, the most general quasi-solvable operator of (at most) second-order which preserves the type C′ space (7.4) is given by the linear combination of all the operators (7.5)–(7.10):

$$\begin{aligned} \tilde{\mathcal{H}}^{(C')} = & - \sum_{\bar{k} \geq \bar{l}}^{M-1} \sum_{\{m_i\}_{\bar{k}+\bar{l}}} A_{\{m_i\}_{\bar{k}+\bar{l}},\bar{k}\bar{l}} F_{\{m_i\}_{\bar{k}+\bar{l}},\bar{k}\bar{l}} - \sum_{\bar{k}=1}^{M-1} \sum_{\{m_i\}_{\bar{k}}} A_{\{m_i\}_{\bar{k}},\bar{k},MM} F_{\{m_i\}_{\bar{k}},\bar{k}} E_{MM} \\ & - \sum_{\{m_i\}_M} A_{\{m_i\}_M,M,MM} F_{\{m_i\}_M,M}(E_{MM} - \lambda) - A_{10,00} F_{10,00} \\ & + \sum_{\bar{k}=1}^{M-1} B_{\{m_i\}_{\bar{k}},\bar{k}} F_{\{m_i\}_{\bar{k}},\bar{k}} + B_{MM} E_{MM} - c_0, \end{aligned} \tag{7.11}$$

where again the coefficients A B with indices and c_0 are real constants and the summation over the set $\{m_i\}_{\bar{k}}$ etc. is understood to take all the possible set of values $\{m_1, \dots, m_M\}$ indicated in Eq. (4.2). In terms of the variables σ, the operator $\tilde{\mathcal{H}}^{(C')}$ is expressed as

$$\begin{aligned} \tilde{\mathcal{H}}^{(C')} = & - \sum_{k,l=1}^{M} \Big[A_{10,00} k l \sigma_1 \sigma_k \sigma_l + \boldsymbol{A}_{kl}(\sigma) \Big] \frac{\partial^2}{\partial \sigma_k \partial \sigma_l} \\ & + \sum_{k=1}^{M} \Big[A_{10,00}(\mathcal{N} + M\lambda - k - 2) k \sigma_1 \sigma_k + \boldsymbol{B}_k(\sigma) \Big] \frac{\partial}{\partial \sigma_k} - \boldsymbol{C}(\sigma), \end{aligned} \tag{7.12}$$

where $\boldsymbol{A}_{kl}$, $\boldsymbol{B}_k$, and $\boldsymbol{C}$ are polynomials of several variables given by

$$\boldsymbol{A}_{\bar{k}\bar{l}}(\sigma) = \sum_{\{m_i\}_{\bar{k}+\bar{l}}} A_{\{m_i\}_{\bar{k}+\bar{l}},\bar{k}\bar{l}} \prod_{i=1}^{M} \sigma_i^{m_i} \quad (\bar{k}, \bar{l} = 1, \dots, M-1;\ \bar{k} \geq \bar{l}), \tag{7.13a}$$

$$\boldsymbol{A}_{Mk}(\sigma) = \sum_{\{m_i\}_k} A_{\{m_i\}_k,k,MM} \prod_{i=1}^{M} \sigma_i^{m_i} \sigma_M \quad (k = 1, \dots, M), \tag{7.13b}$$

$$\boldsymbol{B}_{\bar{k}}(\sigma) = -\sum_{\{m_i\}_{\bar{k}}} B_{\{m_i\}_{\bar{k}},\bar{k}} \prod_{i=1}^{M} \sigma_i^{m_i} \quad (\bar{k} = 1, \dots, M-1), \tag{7.13c}$$

$$\boldsymbol{B}_M(\sigma) = -(\lambda - 1) \sum_{\{m_i\}_M} A_{\{m_i\}_M,M,MM} \prod_{i=1}^{M} \sigma_i^{m_i} - B_{MM}\sigma_M, \tag{7.13d}$$

$$\boldsymbol{C}(\sigma) = (\mathcal{N}_1 - 1)(\mathcal{N}_2 + M\lambda - 1) A_{10,00}\sigma_1 + c_0. \tag{7.13e}$$

Except for the operator $F_{10,00}$, all the operators in Eqs. (7.5)–(7.9) leave the type C′ space (7.4) invariant for *arbitrary* natural numbers $\mathcal{N}_1$ and $\mathcal{N}_2$. Hence, the operator $\tilde{\mathcal{H}}^{(C')}$ is not only quasi-solvable but also solvable if

$$A_{10,00} = 0. \tag{7.14}$$

As in the case of regular type C in Ref. [18], all the operators in Eqs. (7.5)–(7.10) and hence the most general type C′ operator (7.12) preserve *separately* both the subspaces of $\tilde{\mathcal{V}}^{(C')}_{\mathcal{N}_1,\mathcal{N}_2;M}$ in Eq. (7.4), the fact originally comes from the restriction (7.2). In other words, the second-order operators $\sigma_M^{-(k-1)\lambda}\tilde{\mathcal{H}}^{(C')}\sigma_M^{(k-1)\lambda}$ ($k = 1, 2$) leave the type A′ space $\tilde{\mathcal{V}}^{(A')}_{\mathcal{N}_k}$ invariant, respectively. As a consequence, the most general type C′ gauged Hamiltonian $\tilde{H}^{(C')}$ must satisfy the following condition:

$$\tilde{H}^{(C')} = \tilde{H}^{(A')}_{\mathcal{N}_1} = \sigma_M^{\lambda} \tilde{H}^{(A')}_{\mathcal{N}_2} \sigma_M^{-\lambda}. \tag{7.15}$$

In the above, each of the operators $\tilde{H}^{(A')}_{\mathcal{N}_k}$ ($k = 1, 2$) is a type A′ gauged Hamiltonian and thus has the form (5.32) with Eqs. (5.14), (5.28), (5.33), and (5.34). Tracing a completely similar way to Section 5 in Ref. [18] and noting the fact that the first-order operator F_{10} defined by Eq. (4.1b) is missing in the set of the type C′ operators (7.5)–(7.10), we find that the most general type C′ gauged Hamiltonian satisfying the condition (7.15) has the following form:

$$\tilde{H}^{(C')} = -\sum_{i=1}^{M} A(z_i)\frac{\partial^2}{\partial z_i^2} - \sum_{i=1}^{M} B(z_i)\frac{\partial}{\partial z_i} - 2c\sum_{i\neq j}^{M} \frac{A(z_i)}{z_i - z_j}\frac{\partial}{\partial z_i} - c_0, \tag{7.16}$$

where

$$A(z_i) = a_2 z_i^2 + a_1 z_i, \tag{7.17}$$

$$B(z_i) = b_1 z_i - (\lambda - 1) a_1, \tag{7.18}$$

and a_i, b_1, c_0, and c are constants. Comparing the above results with the most general type C gauged Hamiltonian, Eqs. (5.17)–(5.20) in Ref. [18], we see that the type C′ gauged Hamiltonian (7.16) is a special case of the type C gauged Hamiltonian with $a_3 = 0$. Hence, all the quantum mechanical models of type C′ are included in the ones fully classified in Ref. [18].

8 Discussion and Summary

In this article, we have made a new generalization of the type A monomial space of a single variable to several variables and have constructed the most general (at most) second-order quasi-solvable operator which preserves the new linear space called type A′. Examining the condition under which the type A′ second-order operators can be transformed to Schrödinger operators, we have extracted the most general type A′ gauged Hamiltonian. Then, we have completely classified the type A′ quantum Hamiltonians. We have also investigated a new type C generalization called type C′ based on the type A′ space. Combining the results obtained in this article with the ones in Refs. [11,18], we can summarize the classification of the type A′ quasi-solvable quantum many-body systems as shown in Table 2.

Table 2
Classification of the type A′ (quasi-)solvable quantum many-body systems.

Model	Type	Solvable
Rational A Calogero–Sutherland	A′, A	○
+ quadratic M-body interaction	A′	○
Rational BC Calogero–Sutherland	C′, C	○
+ sextic M-body interaction	A′	×
Hyp.(Trig.) A Calogero–Sutherland	C′, C	○
+ external Morse potential	A′, A	○
Hyp.(Trig.) BC Calogero–Sutherland	C′, C	○

The meaning of Table 2 is as follows. For example, the rational BC Calogero–Sutherland model in the third row belongs to type C′ and C, and it is not only quasi-solvable but also solvable. If the sextic M-body interaction is added to

this model (the fourth row), it belongs to type A′ and it is only quasi-solvable but not solvable. The interpretation for other models would be straightforward in a similar way.

One may be curious about the fact that some of the models belong to both type A and A′, or to both type C and C′. Let us first consider the former cases. The fact that a model belongs to both type A and A′ means that it preserves type A and A′ spaces *simultaneously*. In this respect, we note that for fixed values of $M(>1)$ and $\mathcal{N}$ the type A′ space (3.8) is a subspace of the type A space (3.7):

$$\tilde{\mathcal{V}}^{(A')}_{\mathcal{N};M} \subset \tilde{\mathcal{V}}^{(A)}_{\mathcal{N};M}. \tag{8.1}$$

However, it does not mean that an operator which preserves a type A space always preserves a type A′ space too nor mean vice versa. For instance, all the operators of the form

$$E_{ij} = \sigma_i \frac{\partial}{\partial\sigma_j} \quad (i, j = 1, \dots, M;\ i > j), \tag{8.2}$$

preserve type A spaces but do not any type A′ spaces, while the operator F_{10} defined by Eq. (4.1b) preserves the type A′ space with the same $\mathcal{N}$ as in F_{10} but does not any type A spaces. From the set of the operators (4.1) and (4.5)–(4.8) which preserve the type A′ space and the set of operators (3.10) and (3.12) in Ref. [11] which preserve the type A space [4], we find that the set of linearly independent differential operators of (at most) second-order which preserve both the type A and A′ spaces simultaneously is the following:

$$\frac{\partial}{\partial\sigma_i}, \qquad \sigma_i \frac{\partial}{\partial\sigma_j} \quad (i \le j), \tag{8.3a}$$

$$\frac{\partial^2}{\partial\sigma_k \partial\sigma_l}, \quad \sigma_i \frac{\partial^2}{\partial\sigma_k \partial\sigma_l} \quad (i \le k + l), \quad \sigma_i\sigma_j \frac{\partial^2}{\partial\sigma_k \partial\sigma_l} \quad (i + j \le k + l), \tag{8.3b}$$

$$\sigma_i \left(\mathcal{N} - 1 - \sum_{l=1}^{M} \sigma_l \frac{\partial}{\partial\sigma_l} \right) \sigma_j \frac{\partial}{\partial\sigma_k} \quad (i + j \le k). \tag{8.3c}$$

[4] We would like to note that there are typos in Ref. [11]; the following operators are missing in Eqs. (3.12):

$$\frac{\partial^2}{\partial\sigma_i \partial\sigma_j} = E_{0i}E_{0j},$$

and Eq. (3.12c) should be

$$\sigma_i \left(\mathcal{N} - 1 - \sum_{l=1}^{M} \sigma_l \frac{\partial}{\partial\sigma_l} \right) \sigma_j \frac{\partial}{\partial\sigma_k} = E_{i0}E_{jk}.$$

The most general quasi-solvable operator of (at most) second-order $\tilde{\mathcal{H}}^{(A,A')}$ is obviously obtained by the linear combination of all the above operators. In particular, the most general gauged Hamiltonian $\tilde{H}^{(A,A')}$ leaving the type A and A′ spaces simultaneously invariant must be of the form of both the type A gauged Hamiltonian, Eqs. (5.12)–(5.14) in Ref. [11], and the type A′ gauged Hamiltonian, Eqs. (5.32)–(5.34) with (5.14) and (5.28). From the observation, we see that the type A′ gauged Hamiltonian (5.32) belongs to type A too if and only if $g\delta_{a_2,0} = 0$. We note that if it is the case, the system is always solvable since the solvability condition (5.37) is automatically satisfied in the case. In fact, all the models without M-body interactions in Table 2 satisfy the condition $g\delta_{a_2,0} = 0$, thus belong to both type A and A′, and are not only quasi-solvable but also solvable. One of the interesting consequences is that all the models without M-body interactions in Table 2 preserve the following infinite flag of finite dimensional linear spaces:

$$\begin{array}{ccccccc} \mathcal{V}^{(A)}_{1;M} & \subset & \mathcal{V}^{(A)}_{2;M} & \subset \cdots \subset & \mathcal{V}^{(A)}_{\mathcal{N};M} & \subset \cdots \\ \| & & \cup & & \cup & \\ \mathcal{V}^{(A')}_{1;M} & \subset & \mathcal{V}^{(A')}_{2;M} & \subset \cdots \subset & \mathcal{V}^{(A')}_{\mathcal{N};M} & \subset \cdots , \end{array} \tag{8.4}$$

where $\mathcal{V}^{(A)}_{\mathcal{N};M}$ are gauge-transformed type A spaces

$$\mathcal{V}^{(A)}_{\mathcal{N};M} = \mathrm{e}^{-\mathcal{W}}\, \tilde{\mathcal{V}}^{(A)}_{\mathcal{N};M}, \tag{8.5}$$

with the same gauge potential $\mathcal{W}$ as in Eq. (5.38).

The situation in the case of simultaneous type C and C′ is completely analogous to in the case of simultaneous type A and A′ discussed just above. The models which belong to both type C and C′ preserve the type C and C′ spaces simultaneously. From Eqs. (7.3), (7.4), and (8.1), we easily see that for fixed values of $M(> 1)$, $\mathcal{N}_1$, and $\mathcal{N}_2$ the type C′ space is a subspace of the type C space:

$$\tilde{\mathcal{V}}^{(C')}_{\mathcal{N}_1,\mathcal{N}_2;M} \subset \tilde{\mathcal{V}}^{(C)}_{\mathcal{N}_1,\mathcal{N}_2;M}. \tag{8.6}$$

The set of linearly independent differential operators of (at most) second-order which preserve both the type C and C′ spaces simultaneously is the following:

$$\frac{\partial}{\partial\sigma_{\bar{\imath}}},\qquad \sigma_{\bar{\imath}}\frac{\partial}{\partial\sigma_{\bar{\jmath}}}\quad(\bar{\imath}\leq\bar{\jmath}),\qquad \sigma_M\frac{\partial}{\partial\sigma_M},\tag{8.7a}$$

$$\frac{\partial^2}{\partial\sigma_{\bar{k}}\partial\sigma_{\bar{l}}},\quad \sigma_i\frac{\partial^2}{\partial\sigma_{\bar{k}}\partial\sigma_{\bar{l}}}\quad(i\leq\bar{k}+\bar{l}),\quad \sigma_i\sigma_j\frac{\partial^2}{\partial\sigma_{\bar{k}}\partial\sigma_{\bar{l}}}\quad(i+j\leq\bar{k}+\bar{l}),\tag{8.7b}$$

$$\sigma_M\frac{\partial^2}{\partial\sigma_M\partial\sigma_{\bar{k}}},\qquad \sigma_M\sigma_{\bar{\imath}}\frac{\partial^2}{\partial\sigma_M\partial\sigma_{\bar{k}}}\quad(\bar{\imath}\leq\bar{k}),\tag{8.7c}$$

$$\frac{\partial}{\partial\sigma_M}\left(\sigma_M\frac{\partial}{\partial\sigma_M}-\lambda\right),\qquad \sigma_i\frac{\partial}{\partial\sigma_M}\left(\sigma_M\frac{\partial}{\partial\sigma_M}-\lambda\right).\tag{8.7d}$$

The most general quasi-solvable operator of (at most) second-order $\tilde{\mathcal{H}}^{(C,C')}$ is again obviously obtained by the linear combination of all the above operators (8.7). It should be remarked that the operator $\tilde{\mathcal{H}}^{(C,C')}$ is always solvable since all the operators in Eqs. (8.7) preserve the following infinite flag of finite dimensional linear spaces:

$$\begin{array}{ccccccc}
\tilde{\mathcal{V}}^{(C)}_{\mathcal{N}_1,1;M} & \subset & \tilde{\mathcal{V}}^{(C)}_{\mathcal{N}_1,2;M} & \subset\cdots\subset & \tilde{\mathcal{V}}^{(C)}_{\mathcal{N}_1,\mathcal{N}_2;M} & \subset\cdots \\
\cup & & \cup & & \cup & \\
\tilde{\mathcal{V}}^{(C')}_{\mathcal{N}_1,1;M} & \subset & \tilde{\mathcal{V}}^{(C')}_{\mathcal{N}_1,2;M} & \subset\cdots\subset & \tilde{\mathcal{V}}^{(C')}_{\mathcal{N}_1,\mathcal{N}_2;M} & \subset\cdots,
\end{array}\tag{8.8}$$

for all $\mathcal{N}_1=1,2,3,\ldots$. In particular, the type C′ gauged Hamiltonian (7.16) always belongs to type C, as has been mentioned at the end of Section 7, thus always preserves the infinite flag of the spaces (8.8) for all $\mathcal{N}_1=1,2,3,\ldots$. Table 2 indicates that the rational, hyperbolic, trigonometric BC type, and hyperbolic, trigonometric A type Calogero–Sutherland systems have this intriguing property.

Furthermore, it is also interesting to note that the most general operator of (at most) second-order $\tilde{\mathcal{H}}^{(A,A')}$ which preserves both the type A and A′ spaces for a given $\mathcal{N}=n$ always preserves the type A′ spaces for all $\mathcal{N}=1,2,3,\ldots$ but does not the type A spaces for any $\mathcal{N}\neq n$ as far as the operator given by Eq. (8.3c) is included in $\tilde{\mathcal{H}}^{(A,A')}$. As a consequence, the operator $\tilde{\mathcal{H}}^{(A,A')}$ is always solvable but the infinite flag of finite spaces preserved by it has the different structure from Eq. (8.4) as the following:

$$\begin{array}{ccccccccc}
 & & & & \tilde{\mathcal{V}}^{(A)}_{n;M} & & & \\
 & & & & \cup & & & \\
\tilde{\mathcal{V}}^{(A')}_{1;M} & \subset\cdots\subset & \tilde{\mathcal{V}}^{(A')}_{n-1;M} & \subset & \tilde{\mathcal{V}}^{(A')}_{n;M} & \subset & \tilde{\mathcal{V}}^{(A')}_{n+1;M} & \subset\cdots.
\end{array}\tag{8.9}$$

Only when the operator (8.3c) does not exist in $\tilde{\mathcal{H}}^{(A,A')}$, the infinite flag of finite spaces preserved by it has the same structure as Eq. (8.4). The discussion we have made so far surely shows that linear spaces of monomial type which

can be preserved by quantum many-body systems have far richer structure than those preserved by one-body systems. The structure we have revealed in this article would be just the tip of the iceberg.

Finally, we would like to recall the fact that the most general type C′ quasi-solvable operator (7.12) preserves separately the subspaces in the type C′ space (7.4) due to the restriction (7.2). It indicates the existence of *irregular* type C′ operators which do not preserve them separately when the restriction (7.2) is omitted, as in the case of type C [18]. These issues on irregular type C′ together with irregular type C would be reported elsewhere.

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A Formulas

In this appendix, we summarize some useful formulas on the elementary symmetric polynomials.

$$\sum_{k=0}^{M}(-1)^k z_i^{M-k}\sigma_k = \sum_{k=0}^{M}(-z_i)^k\sigma_{M-k} = 0 \quad (i = 1, \ldots, M), \tag{A.1}$$

$$\sum_{k=1}^{M}(-1)^{k+1}k z_i^{M-k-1}\sigma_k = \prod_{j(\neq i)}^{M}(z_i - z_j) \quad (i = 1, \ldots, M), \tag{A.2}$$

$$\frac{\partial\sigma_k}{\partial z_i} = \sum_{l=1}^{k}(-z_i)^{l-1}\sigma_{k-l} \quad (i, k = 1, \ldots, M), \tag{A.3}$$

$$\frac{\partial}{\partial\sigma_k} = \sum_{i=1}^{M}\frac{(-1)^{k+1}z_i^{M-k}}{\prod_{j(\neq i)}^{M}(z_i - z_j)}\frac{\partial}{\partial z_i} \quad (k = 1, \ldots, M). \tag{A.4}$$

The first formula (A.1) is readily derived from the following identity:

$$\prod_{i=1}^{M}(z - z_i) = \sum_{k=0}^{M}(-1)^k z^{M-k}\sigma_k.$$

The second one (A.2) is easily proved inductively. The third one (A.3) is derived from the repeated application of the following formula:

$$\frac{\partial\sigma_k}{\partial z_i} = \sigma_{k-1} - z_i\frac{\partial\sigma_{k-1}}{\partial z_i}.$$

The derivation of the fourth one (A.4) is as follows. Taking the differential of Eq. (A.1), we have

$$\mathrm{d}z_i = \sum_{k=1}^{M} \frac{(-1)^{k+1} z_i^{M-k}}{\sum_{l=0}^{M-1}(-1)^l (M-l) z_i^{M-l-1}\sigma_l} \mathrm{d}\sigma_k \quad (i = 1, \ldots, M).$$

From Eq. (A.1) the denominator in the r.h.s. of the above equation reads,

$$\sum_{l=0}^{M-1}(-1)^l (M-l) z_i^{M-l-1}\sigma_l = \sum_{l=1}^{M}(-1)^{l+1} l z_i^{M-l-1}\sigma_l.$$

Hence, applying Eq. (A.2) we obtain the formula (A.4).

For the derivation of the formulas below, see Appendix B in Ref. [11].

$$\sum_{i=1}^{M} \frac{\partial}{\partial z_i} = \sum_{k=1}^{M}(M-k+1)\sigma_{k-1}\frac{\partial}{\partial \sigma_k}, \tag{A.5a}$$

$$\sum_{i=1}^{M} z_i \frac{\partial}{\partial z_i} = \sum_{k=1}^{M} k\sigma_k \frac{\partial}{\partial \sigma_k}, \tag{A.5b}$$

$$2\sum_{i\neq j}^{M} \frac{1}{z_i - z_j}\frac{\partial}{\partial z_i} = -\sum_{k=1}^{M}(M-k+1)(M-k+2)\sigma_{k-2}\frac{\partial}{\partial \sigma_k}, \tag{A.5c}$$

$$2\sum_{i\neq j}^{M} \frac{z_i}{z_i - z_j}\frac{\partial}{\partial z_i} = -\sum_{k=1}^{M}(M-k)(M-k+1)\sigma_{k-1}\frac{\partial}{\partial \sigma_k}, \tag{A.5d}$$

$$2\sum_{i\neq j}^{M} \frac{z_i^2}{z_i - z_j}\frac{\partial}{\partial z_i} = -\sum_{k=1}^{M} k(2M-k-1)\sigma_k\frac{\partial}{\partial \sigma_k}. \tag{A.5e}$$

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